A PAIRING BETWEEN SUPER LIE-RINEHART AND PERIODIC CYCLIC HOMOLOGY.

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ABSTRACT. We consider a pairing producing various cyclic Hochschild co-cycles, which led Alain Connes to cyclic cohomology. We are interested in geometrical meaning and homological properties of this pairing. We define a non-trivial pairing between the homology of a Lie-Rinehart (super-)algebra with coefficients in some partial traces and relative periodic cyclic homology. This pairing generalizes the index formula for summable Fredholm modules, the Connes-Kubo formula for the Hall conductivity and the formula computing the K^0-group of a smooth noncommutative torus. It also produces new homological invariants of proper maps contracting each orbit contained in a closed invariant subset in a manifold acted on smoothly by a connected Lie group. Finally we compare it with the characteristic map for the Hopf-cyclic cohomology.

1. Introduction. Let G be a simply-connected Lie group acting smoothly on a smooth manifold N and Z be a closed invariant submanifold. Let a smooth map $N \to M$ contract these orbits. In the dual language of algebras of smooth functions the situation can be described as follows. We have a Lie algebra $\mathfrak{g}$ acting by derivations on an algebra B, fixing an ideal J. We have also a homomorphism of algebras $\pi^* : A \to B$ such that $\mathfrak{g}(\pi^*A) \subset J$, or equivalently, we have a homomorphism of algebras

$$(1) \qquad A \to B \times_{B/J} (B/J)^{\mathfrak{g}}.$$

The last homomorphism of algebras of smooth functions describes a continuous map of topological spaces

$$(2) \qquad N \sqcup_P P/G \to M.$$

We generalize this construction to the case of families, parameterized by commutative super-spaces, of noncommutative super-spaces, with the total space acted by super-Lie-Rinehart algebras over the base algebra, as follows. Let (L, R) be a $\mathbb{Z}/2$-graded Lie-Rinehart algebra over a $\mathbb{Z}/2$-graded-commutative ring R containing rational numbers, with a subring of constants $k := H^0(L, R; R) = R^L$, acting (from the left) by super-derivations on a $\mathbb{Z}/2$-graded associative R-algebra B. Provided a homomorphism of $\mathbb{Z}/2$-graded associative k-algebras $A \to B \times_{B/J} (B/J)^L$ is given, we prove the following theorem.

Theorem 1. *There exists a nontrivial canonical k-bilinear pairing*

$$(3) \qquad H_p(L, R; H^0(B, (J^p)^*)) \otimes_k \mathbf{S}(HC_{p+2}(A/k)) \to k.$$

Here $H_p(L, R; -)$ denotes the p-th super-Lie-Rinehart homology [14], [18], $H^0(B, -)$ denotes the 0-th Hochschild cohomology, $(-)^* = Hom_k(-, k)$ and $\mathbf{S} : HC_{p+2}(A/k) \to HC_p(A/k))$ is the periodicity map of Connes on the relative cyclic homology [5]. The above theorem implies, after passing to the inverse limit with respect to $\mathbf{S}$, the existence of a canonical bilinear pairing with the relative periodic cyclic homology of A.

Corollary 1. *There exists a nontrivial canonical bilinear pairing*

$$(4) \qquad H_p(L, R; H^0(B, (J^p)^*)) \otimes_k HP_p(A/k) \to k.$$

The latter pairing induces (is equivalent to, if $k = R$ and k is a field) the following k-linear map

$$(5) \qquad H_p(L, R; H^0(B, (J^p)^*)) \to HP^p(A/k),$$

which can be regarded as a kind of *characteristic map*. We will compare it with the Connes-Moscovici characteristic map [7, 8] and with the cup-product of the second kind of Khalkhali-Rangipour [13] in Hopf-cyclic cohomology.

We will show four classes of examples for which our pairing (or the characteristic map) is known to be, in general, non-trivial and its values have important geometric interpretations. First one is the creation of nontrivial homology classes by contracting orbits making sense in classical differential geometry, the second is the index formula for summable Fredholm modules [5, 6] third is the Connes-Kubo formula for the Hall conductivity in the quantum Hall effect [3, 1, 5, 2, 15, 19], and the fourth computes K^0-group of a noncommutative torus in terms of characteristic numbers of smooth Powers-Rieffel projections [4, 16, 17].

Analogous considerations give us the following "dual" variant, seemingly more fundamental, of our construction for $((L, R), J \subset B)$ and $A \to B$ as above.

Theorem 2. *There exists a nontrivial canonical k-linear "dual characteristic map"*

$$(6) \qquad HP_p(A/k) \to H^p(L, R; H_0(B, J^p)).$$

2. Construction. We consider the bilinear pairing

$$(7) \qquad C_p(L, R; H^0(B, (J^p)^*)) \otimes_k \bigotimes_k^{p+1} A \to k,$$

$$(\tau \otimes X_1 \wedge \ldots \wedge X_p) \otimes (a_0 \otimes \ldots \otimes a_p) \mapsto \sum_{\sigma \in \Sigma_p} (\pm) \tau(a_0 X_{\sigma(1)}(a_1) \ldots X_{\sigma(p)}(a_p)),$$

where the sign is determined uniquely by the convention of transposition of homogeneous symbols from the left hand side to the position on the right hand side. In the sequel we will use homogeneous elements, the above sign convention and the abbreviated notation $X = X_1 \wedge \ldots \wedge X_p \in \bigwedge_R^p L$. Let $\tau \in$

$H^0(B, (J^p)^*) = (J^p/[B, J^p])^*$. The latter space is a right (L, R)-module. The super-Lie-Rinehart boundary operator $\partial : C_p(L, R; -) \to C_{p-1}(L, R; -)$, where $C_p(L, R; -) = (-) \otimes_R \bigwedge_R^p$, computing homology with values in (L, R)-modules is an obvious minimal common generalization of the super-Lie boundary operator from[14] and the Lie-Rinehart boundary operator from [18]. By $Z_p(L, R; -)$ (resp. $B_p(L, R; -)$) we denote cycles (resp. boundaries) in this complex. By $\boldsymbol{b}$ (resp. $\boldsymbol{t}$, $\boldsymbol{B}$) we denote the Hochschild boundary (resp. cyclic operator, Connes B-operator) used in cyclic homology [4]. In the lemmas below we apply the above pairing to various pairs of submodules of super-Lie-Rinehart and Hochschild chains.

Lemma 1.

$$(8) \qquad\qquad C_p(L, R; H^0(B, (J^p)^*)) \cdot \operatorname{im}(\boldsymbol{b}) = 0.$$

   Proof.

$$(9) \qquad\qquad (\tau \otimes X) \cdot \boldsymbol{b}(a_0 \otimes \ldots \otimes a_p) = 0.$$

$\square$

Lemma 2.

$$(10) \qquad\qquad Z_p(L, R; H^0(B, (J^p)^*)) \cdot \operatorname{im}(1 - \boldsymbol{t}) = 0.$$

   Proof.

$$(11) \quad (\tau \otimes X) \cdot (1 - \boldsymbol{t})(a_0 \otimes \ldots \otimes a_p) = \pm \partial(\tau \otimes X) \cdot (a_p a_0 \otimes a_1 \otimes \ldots \otimes a_{p-1}).$$

$\square$

   From the last two lemmas we get

Corollary 2. *There exists a canonical bilinear pairing*

$$(12) \qquad\qquad Z_p(L, R; H^0(B, (J^p)^*)) \otimes HC_p(A/k) \to k.$$

   One could expect that the above pairing descends to Lie algebra homology. But it is not true without an appropriate replacement on the level of cyclic homology.

Lemma 3.

$$(13) \qquad B_p(L, R; H^0(B, (J^p)^*)) \cdot \ker(\boldsymbol{B} : HC_p(A/k) \to HH_{p+1}(A/k)) = 0.$$

   Proof. The following formula is an analog of the Stokes formula

$$(14) \qquad (\tau \otimes X) \cdot \boldsymbol{B}(a_0 \otimes \ldots \otimes a_{p-1}) = \pm p \, \partial(\tau \otimes X) \cdot (a_0 \otimes \ldots \otimes a_{p-1}).$$

$\square$

   By the long exact sequence of Connes

$$(15) \quad \ldots \to HH_{p+2}(A/k) \xrightarrow{\boldsymbol{I}} HC_{p+2}(A/k) \xrightarrow{\boldsymbol{S}} HC_p(A/k) \xrightarrow{\boldsymbol{B}} HH_{p+1}(A/k) \to \ldots$$

we have

$$ker(\boldsymbol{B} : HC_p(A/k) \to HH_{p+1}(A/k)) = im(\boldsymbol{S} : HC_{p+2}(A/k) \to HC_p(A/k)).$$

Together with Lemma 3 and Corollary 2 this gives the pairing

(16) $H_p(L, R; H^0(B, (J^p)^*)) \otimes_k im(\boldsymbol{S} : HC_{p+2}(A/k) \to HC_p(A/k)) \to k$

desired in Theorem 1.

   In order to show that it is non-trivial and interesting we consider the following classes of examples.

3. Example: Contracting orbits. Before we present non-classical examples, we want to explain the classical case in differential topology. Let N be a compact manifold (resp. singular with boundary) acted on by a connected Lie group G with Lie algebra $\mathfrak{g}$, P be a closed invariant subset (resp. containing the singular locus or boundary) and $J \subset B := C^\infty(N)$ be a $\mathfrak{g}$-invariant ideal of smooth functions on N vanishing along P. The action of $\mathfrak{g}$ on differential forms on N is a representation of a $\mathbb{Z}$-graded super-Lie algebra linearly spanned by symbols $(d, \iota_X, \mathcal{L}_X)$, where $X \in \mathfrak{g}$, of degrees $(1, -1, 0)$, subject to the relations

(17) $[d, d] = 0, \ [\iota_X, \iota_Y] = 0, \ [d, \mathcal{L}_X] = 0,$

$$[d, \iota_X] = \mathcal{L}_X, \ [\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}, \ [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}.$$

We will use the following consequence of these relations

(18) $[d, \iota_{X_1} \ldots \iota_{X_p}] =$

$$= \sum_i (-1)^{i-1} \iota_{X_1} \ldots \widehat{\iota_{X_i}} \ldots \iota_{X_p} \mathcal{L}_{X_i} + \sum_{i<j} (-1)^{i+j-1} \iota_{[X_i, X_i]} \iota_{X_1} \ldots \widehat{\iota_{X_i}} \ldots \widehat{\iota_{X_j}} \ldots \iota_{X_p}.$$

Every smooth measure μ on $N \setminus P$, (i.e. a differential top degree form with values in the orientation bundle), such that for every element $f \in J^p$ the product $f\mu$ extends to a smooth measure on the whole N, defines an element

(19) $\int_Y (-)\mu \in H^0(B, (J^p)^*) = (J^p)^*.$

The right $\mathfrak{g}$-action on such element reads as

(20) $(\int_Y (-)\mu) \cdot X = - \int_Y (-)\mathcal{L}_X \mu$

Proposition 1. *If a chain*

(21) $\sum \int_Y (-)\mu \otimes X_1 \wedge \ldots \wedge X_p \in C_p(\mathfrak{g}, H^0(B, (J^p)^*)),$

where μ's are smooth measures on Y as above, is a cycle (resp. a boundary) then the differential form $\sum \iota_{X_1} \ldots \iota_{X_p}\mu$ is closed (resp. exact).

   Proof. The cycle condition for our chain

(22) $\partial \sum \int_Y (-)\mu \otimes X_1 \wedge \ldots \wedge X_p = 0$

is equivalent to

$$(23) \qquad \sum \sum_i (-1)^i X_1 \wedge \ldots \widehat{X_i} \ldots \wedge X_p \otimes \mathcal{L}_{X_i} \mu +$$

$$+ \sum \sum_{i<j} (-1)^{i+j-1} [X_i, X_i] \wedge X_1 \wedge \ldots \widehat{X_i} \ldots \widehat{X_j} \ldots \wedge X_p \otimes \mu = 0,$$

which implies that

$$(24) \qquad \sum \sum_i (-1)^i \iota_{X_1} \ldots \widehat{\iota_{X_i}} \ldots \iota_{X_p} \mathcal{L}_{X_i} \mu +$$

$$+ \sum \sum_{i<j} (-1)^{i+j-1} \iota_{[X_i, X_i]} \iota_{X_1} \ldots \widehat{\iota_{X_i}} \ldots \widehat{\iota_{X_j}} \ldots \iota_{X_p} \mu = 0,$$

which is equivalent to

$$(25) \qquad \sum [d, \iota_{X_1} \ldots \iota_{X_p}] \mu = 0.$$

Since μ is a top degree form $d\mu = 0$, which gives finally

$$(26) \qquad d \sum \iota_{X_1} \ldots \iota_{X_p} \mu = 0.$$

The proof of the implication *"boundary $\Rightarrow$ exact"* is similar. $\square$

Let us consider now a smooth map $\pi : N \to M$ into a compact manifold (resp. singular variety, with boundary) M, contracting each orbit contained in P to a point. Let $A := C^\infty(M)$ be an algebra of smooth functions on M. Then $\mathfrak{g}(\pi^* A) \subset J \subset B$.

Comparing with the canonical map from De Rham homology of currents to periodic cyclic cohomology we see that our characteristic map associates with the Lie homology class of the above cycle a homology class of the closed current j, where

$$(27) \qquad j(\omega) := \sum \int_N (\pi^* \omega)(\tilde{X}_1, \ldots, \tilde{X}_p) \mu = \pm \int_N (\pi^* \omega) \wedge \sum \iota_{X_1} \ldots \iota_{X_p} \mu.$$

Here by $\tilde{X}$ we mean the vector field corresponding to an element $X \in \mathfrak{g}$.

Note that if $\sum \iota_{X_1} \ldots \iota_{X_p} \mu$ extends to the whole N then we get the push-forward of the homology class of the closed current

$$(28) \qquad \left[\sum \int_N (-) \mu \otimes X_1 \wedge \ldots \wedge X_p \right] \mapsto \pm \pi_! \left[\sum \int_N (-) \iota_{X_1} \ldots \iota_{X_p} \mu \right]$$

in periodic cyclic cohomology.

Take for example $N = M = G = S^1$ and $\pi = \mathrm{id} : S^1 \to S^1$, $P = \emptyset$, μ the Haar measure and $X \in \mathfrak{g}$ normalized so that $\iota_X \mu = 1$. Then $\int_{S^1} (-) \mu \otimes X$ is a cycle and the characteristic map gives

$$(29) \qquad \left[\int_{S^1} (-) \mu \otimes X \right] \mapsto \left[\int_{S^1} \right],$$

i.e. the fundamental class of S^1. Though it is a nontrivial example, it is not very enlightening. Therefore we need more complicated example to show the point. This time $\sum \iota_{X_1} \ldots \iota_{X_p}\mu$ will not extend to the whole N, so the push-forward of the respective current will not be defined. However, our characteristic map still will define a nontrivial homology class on M. To see this, let us take a cylinder $N = S^1 \times [-\pi, \pi]$ with coordinates (φ, ψ) (φ - circular coordinate, $\psi \in [-\pi, \pi]$). Consider the following smooth action of the additive Lie group $G = \mathbb{R}$ on N

$$t \cdot (\varphi, \psi) := \begin{cases} (\varphi + t, 2\arctan(\tan\frac{\psi}{2} + t)) & \text{if } \psi \neq \pm\pi, \\ (\varphi + t, \pm\pi)) & \text{if } \psi = \pm\pi. \end{cases}$$

It is generated by a vector field

$$X = \frac{\partial}{\partial\varphi} + (1 + \cos\psi)\frac{\partial}{\partial\psi}.$$

Let us take $P = \partial N$, which is the union of compact orbits of the above action and the ideal $J = (1 + \cos\psi)$ in the algebra $B = C^\infty(N)$, vanishing along P. The form

$$\mu = \frac{1}{2\pi}\frac{d\psi \wedge d\varphi}{1 + \cos\psi}$$

is defined on $N \setminus P$ and invariant. We have

$$\iota_X\mu = \frac{1}{2\pi}d\varphi - \frac{1}{2\pi}\frac{d\psi}{1 + \cos\psi},$$

which does not extend onto the whole N.

Take now a subvariety $M \subset \mathbb{R}^3$

$$z^2 = \varepsilon^2(\frac{x}{\sqrt{x^2 + y^2}} + 1)^2 - (\sqrt{x^2 + y^2} - 1)^2, \quad (x, y) \neq (0, 0),$$

where a parameter $\varepsilon \in (0, 1/2)$, homeomorphic to a torus with one basic cycle contracted to the unique singular point $(-1, 0, 0)$, and a smooth map $\pi : N \to M$ of the form

$$\pi^*x = \cos\psi(1 + \varepsilon(1 + \cos\psi)\cos\varphi),$$
$$\pi^*y = \sin\psi(1 + \varepsilon(1 + \cos\psi)\cos\varphi),$$
$$\pi^*z = \varepsilon(1 + \cos\psi)\sin\varphi.$$

One can check that $X(\pi^*x), X(\pi^*y), X(\pi^*z) \in J$ which implies that for $A = C^\infty(M)$

$$X(\pi^*A) \subset J.$$

We have $H_1(N, \mathbb{Z}) = \mathbb{Z}$ generated by the homology class of one boundary circle ($\psi = \pi$), $H_1(M, \mathbb{Z}) = \mathbb{Z}$ generated by the homology class of the ellipse ($x^2 + y^2 = 1, z = \varepsilon(x + 1)$). The topology of the map $\pi : N \to M$ is following. It contracts the boundary circles of the cylinder N to the singular point of M. Therefore it kills the generator of $H_1(N, \mathbb{Z})$. But it also creates the generator of

$H_1(M, \mathbb{Z})$. The killing property of π is described by the nullity of the induced map $H_1(N, \mathbb{Z}) \to H_1(M, \mathbb{Z})$. We will show that the creating property of π is described by our characteristic map. Let

$$\omega = \frac{1}{2\pi} \frac{xdy - ydx}{x^2 + y^2}$$

be a closed 1-form on M, whose period over the generator of $H_1(M, \mathbb{Z})$ is equal to 1. We have $\pi^*\omega = d\psi$. Let us compute our pairing of the Lie homology class with the De Rham cohomology class of ω

$$\left[\int_N (-)\mu \otimes X \right] \cdot [\omega] = - \int_N (\pi^*\omega) \wedge \iota_X \mu = -\frac{1}{4\pi^2} \int_N d\psi \wedge d\varphi = 1.$$

Therefore our characteristic map applied to $\left[\int_N (-)\mu \otimes X \right]$ gives the homology class of the current homological to the period over the generator of $H_1(M, \mathbb{Z})$.

4. **Example: Index formula**. Let us assume that $k = \mathbb{C}$ and we have a p-summable even involutive Fredholm module (A, H, F) [5, 6], i.e. A is a $\mathbb{Z}/2$-graded $*$-algebra, H is a $\mathbb{Z}/2$-graded Hilbert space with a grading preserving $*$-representation $A \to B(H)$, and F is an odd self-adjoint involution on H such that

(30) $$[F, A] \subset L^p(H),$$

where $L^p(H)$ denotes the p-th Schatten ideal in $B(H)$.

Let us define now a $\mathbb{Z}/2$-graded abelian super-Lie algebra generated by one odd element d

(31) $$\mathfrak{g} := \mathbb{C} \cdot d,$$

a $\mathbb{Z}/2$-graded associative algebra

(32) $$B := B(H)$$

and an ideal

(33) $$J := L^p(H).$$

The formula

(34) $$db := [F, b]$$

defines the left action of $\mathfrak{g}$ on B by derivations and obviously J is a $\mathfrak{g}$-ideal.

The projection into the first cartesian factor defines an isomorphism of $\mathbb{Z}/2$-graded associative algebras

(35) $$B \times_{B/J} (B/J)^{\mathfrak{g}} \xrightarrow{\cong} \{ b \in B(H) \mid [F, b] \in L^p(H) \}$$

and a $*$-homomorphism of $\mathbb{Z}/2$-graded associative C^*-algebras $A \to B \times_{B/J} (B/J)^{\mathfrak{g}}$ is equivalent to a structure of p-summable even involutive Fredholm module (A, H, F). By functoriality of Connes' long exact sequence it is enough to consider our pairing for $A := B \times_{B/J} (B/J)^{\mathfrak{g}}$.

Since the super-trace is a linear functional on the p-th power of the ideal $J := L^p(H)$ which vanishes on super-commutators and the super-Lie algebra $\mathfrak{g} = \mathbb{C} \cdot d$ is abelian, the element $\mathrm{str} \otimes d \wedge \ldots \wedge d \in H^0(A, (J^p)^*) \otimes \bigwedge^p \mathfrak{g}$ is a cycle. On the other hand, for any even self-adjoint idempotent $e \in A$ the element $e \otimes \ldots \otimes e \in \bigotimes^{p+1} A$ is a cyclic cycle for even p, which is in the image of the periodicity operator $\boldsymbol{S}$. We can compute our pairing of homology classes of these two cycles which gives the index of a Fredholm operator

$$(36) \qquad [\mathrm{str} \otimes d \wedge \ldots \wedge d] \cdot [e \otimes \ldots \otimes e] = c_p \, \mathrm{Index}(e_{11} F_{01} e_{00}),$$

in general a non-zero number. Here $F_{01} : H_0 \to H_1$ (resp. $e_{00} : H_0 \to H_0$, $e_{11} : H_1 \to H_1$) is a unitary block of F (resp. self-adjoint idempotent block of e) under the orthogonal decomposition $H = H_0 \oplus H_1$ into even and odd part.

5. Example: Connes-Kubo formula. Let $\mathfrak{g}$ be an abelian Lie algebra and $A = B = J$. If τ is a $\mathfrak{g}$-invariant trace on A then this is obvious that for all $X_1, \ldots, X_p \in \mathfrak{g}$ the chain

$$(37) \qquad \tau \otimes X_1 \wedge \ldots \wedge X_p$$

is a cycle hence defines a homology class. This construction is next adapted to the geometry of the Brillouin zone. Its pairing with an appropriately normalized even dimensional class $[e \otimes e \otimes e]$ in $HP_2(A)$ computes the Hall conductivity σ in noncommutative geometric models of quantum Hall effect

$$(38) \qquad [\tau \otimes \sum_{i=1}^{g} X_i \wedge X_{i+g}] \cdot [e \otimes e \otimes e] = \sigma,$$

in general a non-zero integer [3, 1, 5, 2, 19] or rational number [15], depending on the model.

6. Example: K_0 of a noncommutative torus. Formally it is the same construction as in the previous example adapted to the context of non-commutative geometry of the noncommutative 2-torus [17]. Let A be the dense subalgebra of "smooth functions on the noncommutative 2-torus" [4] of the C^*-algebra generated by two unitaries U, V subject to the relation

$$(39) \qquad UV = e^{2\pi i \theta} VU$$

with an irrational real θ. The Lie group $S^1 \times S^1$ acts on A by automorphisms and its Lie algebra $\mathfrak{g}$ spanned by commuting elements X, Y acts by derivations such that

$$(40) \qquad X(U) = 2\pi i U, \ X(V) = 0,$$

$$(41) \qquad Y(U) = 0, \ Y(V) = 2\pi i V.$$

Every element $a \in A$ can be uniquely expanded as $a = \sum a_{mn} U^m V^n$. Then the functional $\tau(a) = a_{00}$ is a $\mathfrak{g}$-invariant trace. Again, we have a homology class

$[\tau \otimes X \wedge Y]$. It is known that $K_0(A) = \mathbb{Z} + \mathbb{Z} \cdot \theta \subset \mathbb{C}$ where the identification is done by this trace [16, 17]. Any selfadjoint idempotent $e \in A$ is determined by its trace $\tau(e) = p - q \cdot \theta$ uniquely up to unitary equivalence. This is our pairing in dimension zero

$$(42) \qquad\qquad [\tau] \cdot [e] = p - q \cdot \theta.$$

Our pairing in dimension two computes the number q

$$(43) \qquad\qquad [\tau \otimes X \wedge Y] \cdot [e \otimes e \otimes e] = q \cdot 2\pi i.$$

This means that our pairings, defined *a priori* over $\mathbb{C}$, detect fully the K_0-group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

7. Comparison with other constructions.
In [7, 8] the following pairing (Connes-Moscovici characteristic map) is considered

$$(44) \qquad\qquad HP^p_{(\delta,\sigma)}(H) \otimes Tr_{(\delta,\sigma)}(A) \to HP^p(A)$$

for any Hopf algebra H with (δ, σ) a modular pair in involution, acting on an algebra A, where by $Tr_{(\delta,\sigma)}(A)$ we denote the space of (δ, σ)-traces on A. Taking $H = U(\mathfrak{g})$, $\delta = \epsilon$, $\sigma = 1$ one has [8]

$$(45) \qquad\qquad HP^p_{(\epsilon,1)}(U(\mathfrak{g})) = \bigoplus_{i \equiv p \pmod 2} H_i(\mathfrak{g}),$$

$$(46) \qquad\qquad Tr_{(\epsilon,1)}(A) = H^0(A, A^*)^{\mathfrak{g}}.$$

Then we have the following commuting diagram

$$(47) \qquad \begin{array}{ccc} H_p(\mathfrak{g}) \otimes H^0(A, A^*)^{\mathfrak{g}} & \to & H_p(\mathfrak{g}, H^0(A, A^*)) \\ \downarrow & & \downarrow \\ HP^p_{(\epsilon,1)}(U(\mathfrak{g})) \otimes Tr_{(\epsilon,1)}(A) & \to & HP^p(A), \end{array}$$

where left vertical and upper horizontal arrows are canonical, the bottom horizontal arrow is the Connes-Moscovici characteristic map and the right vertical arrow is our characteristic map for $A = B = J$. The main difference between these two characteristic maps is the position of traces: in the Connes-Moscovici map traces are paired with cyclic periodic cohomology while in our map they are coefficients of Lie algebra homology.

Recently [13] a new pairing with values in cyclic cohomology (the cup product of the second kind)

$$(48) \qquad\qquad HC^p_H(C, M) \otimes HC^q_H(A, M) \to HC^{p+q}(A)$$

has been presented, which allows to consider in this pairing cyclic cohomology with nontrivial coefficients in the sense of [9]. It is defined for a Hopf algebra H, an H-module algebra A, an H-comodule algebra B, an H-module coalgebra C acting on A in a suitable sense and any stable anti-Yetter-Drinfeld (SAYD) module M over H. For $C = H = U(\mathfrak{g})$, $q = 0$ and $M = k_{(\epsilon,1)}$ trivial one

dimensional SAYD-module one gets again the Connes-Moscovici characteristic map.

Since $U(\mathfrak{g})$ is a cocommutative Hopf algebra, any $U(\mathfrak{g})$-module M with a trivial $U(\mathfrak{g})$-comodule structure is a SAYD-module. Taking $C = H = U(\mathfrak{g})$, $q = 0$ and $M = H^0(A, A^*)$ one has the Khalkhali-Rangipour cup product of the second kind

$$(49) \qquad HC^p_{U(\mathfrak{g})}(U(\mathfrak{g}), H^0(A, A^*)) \otimes HC^0_{U(\mathfrak{g})}(A, H^0(A, A^*)) \to HC^p(A).$$

The trace evaluation map $H^0(A, A^*) \otimes A \to k$ defines a distinguished element in $HC^0_{U(\mathfrak{g})}(A, H^0(A, A^*))$ and consequently the following characteristic map

$$(50) \qquad\qquad HC^p_{U(\mathfrak{g})}(U(\mathfrak{g}), H^0(A, A^*)) \to HC^p(A),$$

by taking the above cup product with this distinguished element. In fact this map comes from the morphism of cyclic objects, so it can be pushed to the periodic cyclic cohomology, hence we get a map

$$(51) \qquad\qquad HP^p_{U(\mathfrak{g})}(U(\mathfrak{g}), H^0(A, A^*)) \to HP^p(A).$$

By Theorem 5.2 of [10] (note that in [10] authors use cyclic objects related to cyclic objects from [9] by Connes's cyclic duality [12], transforming homology into cohomology) one has

$$(52) \qquad HP^p_{U(\mathfrak{g})}(U(\mathfrak{g}), H^0(A, A^*)) = \bigoplus_{i \equiv p \ (mod \ 2)} H_i(\mathfrak{g}, H^0(A, A^*)).$$

In this case, our characteristic map factorizes through (51)

$$(53) \qquad\qquad \begin{array}{ccc} & & H_p(\mathfrak{g}, H^0(A, A^*)) \\ & \nearrow & \downarrow \\ HP^p_{U(\mathfrak{g})}(U(\mathfrak{g}), H^0(A, A^*)) & \to & HP^p(A), \end{array}$$

where the south-west arrow is an embedding onto a direct summand in the decomposition (52).

However, in general, there is no way to extend a partial trace from the ideal J^p to the trace defined on the whole algebra, hence there is no a canonical element to pair with as in (50). Therefore the characteristic map á la Khalkhali-Rangipour "$HP^p_{U(\mathfrak{g})}(U(\mathfrak{g}), H^0(B, (J^p)^*)) \to HP^p(A)$" is not defined in general. In particular, the index pairing discussed in the paragraph 4 cannot be obtained in this way.

In spite of this discrepancy we expect that, after appropriate modifications of Hopf-cyclic cohomology (or its extended version [11] working in the case of enveloping algebras of Lie-Rinehart algebras) with appropriate coefficients, our construction could be generalized to (super, extended) Hopf-cyclic cohomology. The crucial property this generalization should satisfy is the above index pairing.

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